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Higher even-order convergence and coupled solutions for second-order boundary value problems on time scales☆

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Abstract

In this paper, for a second-order boundary value problem on time scales, a method of generalized quasilinearization, under coupled upper and lower solutions, is discussed. An attempt for the method is to establish sufficient conditions for generating monotone iterative schemes whose elements converge rapidly to the unique solution of the given problem. Furthermore, the convergence is of order k ($k \geq 2$), which is even. Finally, two examples are provided to illustrate our results.

Keywords: Time scales; Boundary value problem; Quasilinearization; Quadratic convergence; Coupled upper and lower solutions

1. Introduction

The study of dynamical systems on time scales is now an active field of research as it unifies existing discussions in differential and finite difference systems, and provides powerful new tools for exploring connections between the traditionally separated fields. The pioneering works in this field include those in references [1–3]. On the other hand, the method of quasilinearization combined with the method of upper and lower solutions is an effective and fruitful technique for a wide variety of boundary value problems for nonlinear differential systems (see [4–8]), as the technique can be used to establish a sequence of approximate solutions that converges rapidly to the solutions of the given systems. The method has further been extended over the past years and referred to as a method of generalized quasilinearization, and is extremely useful in scientific computations due to its accelerated rate of convergence as in [9,10].

At present, the method has also been successfully applied to dynamical systems on time scales [11]. Recently, Akin [12] has initiated the study of upper and lower solutions to boundary value problems (BVP) on time scales. Subsequently, in [13–15], the method of quasilinearization has been developed for BVP involving the natural upper

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and lower solutions. More recently, Akin-Bohner and Atici [16] studied the method of quasilinearization, under the notion of upper and lower solution, for the BVP

$$\begin{aligned} &-(p(t)x^{\Delta})^{\Delta} + q(t)x^{\sigma} = f(t, x^{\sigma}) + g(t, x^{\sigma}), \quad t \in [a, b]^{k^2}, \\ &x(a) = A, \quad x(b) = B, \end{aligned} \tag{1.1}$$

where $f^{(1)}(t, x) \leq 0$, $f^{(i)}(t, x) \leq 0$ $(2 \leq i \leq k)$, $f^{(k+1)}(t, x) \geq 0$ and $g^{(i)}(t, x) \leq 0$ $(1 \leq i \leq k+1)$ on $[a, b]^{k^2} \times [\alpha_0, \beta_0]$.

In this paper, we should consider a method of generalized quasilinearization, with even-order k $(k \geq 2)$ convergence, for the BVP

$$\begin{aligned} &-(p(t)x^{\Delta})^{\nabla} + q(t)x^{\sigma} = f(t, x^{\sigma}) + g(t, x^{\sigma}), \quad t \in [a, b], \\ &\tau_1 x(\rho(a)) - \tau_2 x^{\Delta}(\rho(a)) = 0, \qquad x(\sigma(b)) - \tau_3 x(\eta) = 0. \end{aligned} \tag{1.2}$$

An attempt here is, including a more general concept of upper and lower solution [17] in mathematical biology, to relax the monotone conditions on $f^{(i)}(t, x)$ and $g^{(i)}(t, x)$ $(1 < i < k)$, so that the high-order convergence of the iterations is ensured for a larger class of nonlinear functions on time scales.

Consequently, the paper is organized in the following manner. In Section 2, we should introduce the coupled upper and lower solutions of the BVP (1.2), and their fundamental properties. In Section 2, sufficient conditions for a method of generalized quasilinearization concerning coupled upper and lower solutions of Type I, are given for the BVP (1.2). In Section 4, we only state without proof, the results obtained similar to those in Sections 2 and 3, for coupled upper and lower solution of Type II of the BVP (1.2), and two examples are added to illustrate the results obtained.

2. Preliminaries

Consider the boundary value problem (BVP)

$$-(p(t)x^{\Delta})^{\nabla} + q(t)x^{\sigma} = f(t, x^{\sigma}) + g(t, x^{\sigma}), \quad t \in [a, b], \tag{2.1}$$

$$\tau_1 x(\rho(a)) - \tau_2 x^{\Delta}(\rho(a)) = 0, \qquad x(\sigma(b)) - \tau_3 x(\eta) = 0, \tag{2.2}$$

where $\sigma(b) > 0$, $\tau_1, \tau_2 \geq 0$, $\tau_1 + \tau_2 > 0$, $0 < \tau_3 < 1$, $\eta \in (\rho(a), \sigma(b))$; $f, g \in C([a, b] \times R)$ and $p, q \in C_{rd}([a, b])$ such that $p(t) > 0$ and $q(t) \geq 0$ on $[a, b]$. Again, let $\mathbb{T}$ be any time scales and $[a, b]$ be a subset of $\mathbb{T}$ such that $[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}$. The details of basic notions connected to time scales are omitted here and we refer the reader to [1]. We define the set

$$\mathbb{D} = \{x \in \mathbb{B} : x^{\Delta} \text{ is continuous on } [\rho(a), \sigma(b)]^k,\ px^{\Delta} \text{ is nabla-differentiable and } (px^{\Delta})^{\nabla} \text{ is continuous on } [\rho(a), \sigma(b)]_k^k\},$$

where the Banach space $\mathbb{B} = C([\rho(a), \sigma(b)])$ is the set of real-valued continuous (in the topology $\mathbb{T}$) functions $x(t)$ defined on $[\rho(a), \sigma(b)]$ with the norm

$$\|x\| = \max_{t \in [\rho(a), \sigma(b)]} |x(t)|.$$

Definition 2.1. With respect to problem (2.1) and (2.2), real-valued functions $\alpha(t), \beta(t) \in \mathbb{D}$, on $[\rho(a), \sigma(b)]$, are called

(i) coupled lower and upper solutions of Type I, if the following inequalities hold

$$\begin{aligned} &-(p(t)\alpha^{\Delta})^{\nabla} + q(t)\alpha^{\sigma} \leq f(t, \alpha^{\sigma}) + g(t, \beta^{\sigma}), \quad t \in [a, b], \\ &\tau_1 \alpha(\rho(a)) - \tau_2 \alpha^{\Delta}(\rho(a)) \leq 0, \qquad \alpha(\sigma(b)) \leq \tau_3 \alpha(\eta); \\ &-(p(t)\beta^{\Delta})^{\nabla} + q(t)\beta^{\sigma} \geq f(t, \beta^{\sigma}) + g(t, \alpha^{\sigma}), \quad t \in [a, b], \\ &\tau_1 \beta(\rho(a)) - \tau_2 \beta^{\Delta}(\rho(a)) \geq 0, \qquad \beta(\sigma(b)) \geq \tau_3 \beta(\eta). \end{aligned}$$

(ii) coupled lower and upper solutions of Type II, if the following inequalities hold

$$\begin{aligned} &-(p(t)\alpha^{\Delta})^{\nabla} + q(t)\alpha^{\sigma} \le f(t,\beta^{\sigma}) + g(t,\alpha^{\sigma}), \quad t \in [a,b], \\ &\tau_1\alpha(\rho(a)) - \tau_2\alpha^{\Delta}(\rho(a)) \le 0, \qquad \alpha(\sigma(b)) \le \tau_3\alpha(\eta); \\ &-(p(t)\beta^{\Delta})^{\nabla} + q(t)\beta^{\sigma} \ge f(t,\alpha^{\sigma}) + g(t,\beta^{\sigma}), \quad t \in [a,b], \\ &\tau_1\beta(\rho(a)) - \tau_2\beta^{\Delta}(\rho(a)) \ge 0, \qquad \beta(\sigma(b)) \ge \tau_3\beta(\eta). \end{aligned}$$

Remark 2.1. The smoothness requirements on $\alpha(t)$ and $\beta(t)$ are weakened in comparison with those in [12].

Lemma 2.1 (*See [12]*). *Assume $h \in \mathbb{D}$. Choose $c \in (a,b)$ such that*

$$h(c) = \max\{h(t) : t \in [a,b]\} \quad \text{and} \quad h(t) < h(c), \quad \text{for } t \in (c,b].$$

Then $h^{\Delta}(c) \le 0$ and $(ph^{\Delta})^{\nabla}(\rho(c)) \le 0$.

Theorem 2.1. *For the BVP (2.1) and (2.2), assume that*

(i) *$\alpha(t)$ and $\beta(t)$ are coupled lower and upper solutions of Type I for (2.1) and (2.2), on $[\rho(a),\sigma(b)]$, respectively;*
(ii) *$f(t,x)$ is strictly decreasing in x and $g(t,x)$ is increasing in x for $t \in [a,b]$.*

Then $\alpha(t) \le \beta(t)$ on $[\rho(a),\sigma(b)]$.

Proof. Define $h = \alpha - \beta$. For the sake of contradiction, let us assume the result is false, then we can find $c \in [a,b]$ such that $h(t)$ has a positive maximum at $t = c$, that is, c satisfies the hypotheses of Lemma 2.1. Thus, we have $(ph^{\Delta})^{\nabla}(\rho(c)) \le 0$. On the other hand, note that $\sigma(\rho(c)) = c$ and in view of hypotheses (i) and (ii), we have

$$\begin{aligned} (ph^{\Delta})^{\nabla}(\rho(c)) &= (p\alpha^{\Delta})^{\nabla}(\rho(c)) - (p\beta^{\Delta})^{\nabla}(\rho(c)) \\ &\ge -f(\rho(c),\alpha(c)) - g(\rho(c),\beta(c)) + f(\rho(c),\beta(c)) + g(\rho(c),\alpha(c)) + q(\rho(c))[\alpha(c)-\beta(c)] \\ &> 0, \end{aligned}$$

which is a contradiction. □

Theorem 2.2. *Assume that $\alpha(t)$, $\beta(t)$ are coupled lower and upper solutions of Type I or Type II for the BVP (2.1) and (2.2), respectively, such that $\alpha(t) \le \beta(t)$ on $[\rho(a),\sigma(b)]$. Then, there exists a solution $x(t)$ of the BVP (2.1) and (2.2), such that $\alpha(t) \le x(t) \le \beta(t)$ on $[\rho(a),\sigma(b)]$.*

Proof. The proof follows by the repeated application of Theorem 2.1 and the details are omitted to avoid repetition. □

Corollary 2.1. *Under the hypotheses of Theorem 2.1, the solution of the BVP (2.1) and (2.2) is unique.*

3. Main results

Firstly, we denote the sector for every $u, v \in \mathbb{D}$, such that

$$[u,v] = \{w \in \mathbb{D} : u(t) \le w(t) \le v(t), t \in [\rho(a),\sigma(b)]\}.$$

In the following results, note that f_x, f_{xx} are the usual partial derivatives of f over the time scales R.

Theorem 3.1. *For the BVP (2.1) and (2.2), in addition to the hypotheses of Theorem 2.1, assume that*

(A1) *$\alpha_0(t)$ and $\beta_0(t)$ are coupled lower and upper solutions of Type I for the BVP (2.1) and (2.2), respectively, such that $\alpha_0(t) \le \beta_0(t)$ on $[\rho(a),\sigma(b)]$;*
(A2) *$f, g \in C^2([\rho(a),\sigma(b)] \times [\alpha_0,\beta_0])$, satisfy*

$$f_{xx}(t,x) \ge 0, \qquad g_{xx}(t,x) \le 0 \quad \text{for } (t,x) \in [\rho(a),\sigma(b)] \times [\alpha_0,\beta_0].$$

Then there exist monotone sequences, $\{\alpha_n\}$ and $\{\beta_n\}$, converging uniformly and quadratically on $[\rho(a), \sigma(b)]$ to the unique solution of the BVP (2.1) and (2.2).

Proof. Define F and G by

$$F(t, x^\sigma, y^\sigma; \alpha_0, \beta_0) = f(t, \alpha_0^\sigma) + g(t, \beta_0^\sigma) + f_x(t, \alpha_0^\sigma)\,(x^\sigma - \alpha_0^\sigma) + g_x(t, \alpha_0^\sigma)\,(y^\sigma - \beta_0^\sigma),$$
$$G(t, x^\sigma, y^\sigma; \alpha_0, \beta_0) = f(t, \beta_0^\sigma) + g(t, \alpha_0^\sigma) + f_x(t, \alpha_0^\sigma)\,(x^\sigma - \beta_0^\sigma) + g_x(t, \alpha_0^\sigma)\,(y^\sigma - \alpha_0^\sigma). \quad \square$$

In addition to the BVP (2.1) and (2.2), we also consider the following BVP

$$-(p(t)x^\Delta)^\nabla + q(t)x^\sigma = F(t, x^\sigma, y^\sigma; \alpha_0, \beta_0), \quad t \in [a, b], \tag{3.1}$$

subject to the boundary conditions (2.2) and

$$-(p(t)x^\Delta)^\nabla + q(t)x^\sigma = G(t, x^\sigma, y^\sigma; \alpha_0, \beta_0), \quad t \in [a, b], \tag{3.2}$$

subject to the boundary conditions (2.2).

Applying Taylor's theorem and hypotheses (A1), (A2) implies

$$F(t, \alpha_0^\sigma, \beta_0^\sigma; \alpha_0, \beta_0) = f(t, \alpha_0^\sigma) + g(t, \beta_0^\sigma) \ge -(p(t)\alpha_0^\Delta)^\nabla + q(t)\alpha_0^\sigma,$$
$$\begin{aligned} F(t, \beta_0^\sigma, \alpha_0^\sigma; \alpha_0, \beta_0) &= f(t, \alpha_0^\sigma) + g(t, \beta_0^\sigma) + f_x(t, \alpha_0^\sigma)(\beta_0^\sigma - \alpha_0^\sigma) + g_x(t, \alpha_0^\sigma)\,(\alpha_0^\sigma - \beta_0^\sigma) \\ &= f(t, \alpha_0^\sigma) + f_x(t, \alpha_0^\sigma)(\beta_0^\sigma - \alpha_0^\sigma) + g(t, \beta_0^\sigma) + g_x(t, \alpha_0^\sigma)(\alpha_0^\sigma - \beta_0^\sigma) \\ &= f(t, \beta_0^\sigma) - \frac{1}{2!} f_{xx}(t, c_1)\,(\beta_0^\sigma - \alpha_0^\sigma)^2 + g(t, \alpha_0^\sigma) + \frac{1}{2!} g_{xx}(t, c_2)\,(\beta_0^\sigma - \alpha_0^\sigma)^2 \\ &\le f(t, \beta_0^\sigma) + g(t, \alpha_0^\sigma) \le -(p(t)\beta_0^\Delta)^\nabla + q(t)\beta_0^\sigma, \end{aligned}$$

where $\alpha_0^\sigma \le c_1, c_2 \le \beta_0^\sigma$.

Hence by Theorem 2.2 there exists a solution $\alpha_1(t)$ of the BVP (3.1) and (2.2) such that

$$\alpha_0(t) \le \alpha_1(t) \le \beta_0(t), \quad t \in [\rho(a), \sigma(b)].$$

Similarly, using Taylor's theorem and hypotheses (A1)–(A2), we obtain

$$\begin{aligned} G(t, \alpha_0^\sigma, \beta_0^\sigma; \alpha_0, \beta_0) &= f(t, \beta_0^\sigma) + g(t, \alpha_0^\sigma) + f_x(t, \alpha_0^\sigma)\,(\alpha_0^\sigma - \beta_0^\sigma) + g_x(t, \alpha_0^\sigma)\,(\beta_0^\sigma - \alpha_0^\sigma) \\ &= f(t, \beta_0^\sigma) - f_x(t, \alpha_0^\sigma)\,(\beta_0^\sigma - \alpha_0^\sigma) + g(t, \alpha_0^\sigma) + g_x(t, \alpha_0^\sigma)\,(\beta_0^\sigma - \alpha_0^\sigma) \\ &= f(t, \alpha_0^\sigma) + \frac{1}{2!} f_{xx}(t, c_1)\,(\beta_0^\sigma - \alpha_0^\sigma)^2 + g(t, \beta_0^\sigma) - \frac{1}{2!} g_{xx}(t, c_2)\,(\beta_0^\sigma - \alpha_0^\sigma)^2 \\ &\ge f(t, \alpha_0^\sigma) + g(t, \beta_0^\sigma) \ge -(p(t)\alpha_0^\Delta)^\nabla + q(t)\alpha_0^\sigma, \end{aligned}$$
$$G(t, \beta_0^\sigma, \alpha_0^\sigma; \alpha_0, \beta_0) = f(t, \beta_0^\sigma) + g(t, \alpha_0^\sigma) \le -(p(t)\beta_0^\Delta)^\nabla + q(t)\beta_0^\sigma,$$

where $\alpha_0^\sigma \le c_1, c_2 \le \beta_0^\sigma$, and therefore, as before, there exists a solution $\beta_1(t)$ of (3.2) and (2.2) such that

$$\alpha_0(t) \le \beta_1(t) \le \beta_0(t), \quad t \in [\rho(a), \sigma(b)].$$

Now since $-(p(t)\alpha_1^\Delta)^\nabla + q(t)\alpha_1^\sigma = F(t, \alpha_1^\sigma, \beta_1^\sigma; \alpha_0, \beta_0)$, we get that

$$\begin{aligned} -(p(t)\alpha_1^\Delta)^\nabla + q(t)\alpha_1^\sigma &= f(t, \alpha_0^\sigma) + g(t, \beta_0^\sigma) + f_x(t, \alpha_0^\sigma)\,(\alpha_1^\sigma - \alpha_0^\sigma) + g_x(t, \alpha_0^\sigma)\,(\beta_1^\sigma - \beta_0^\sigma) \\ &= f(t, \alpha_0^\sigma) + f_x(t, \alpha_0^\sigma)\,(\alpha_1^\sigma - \alpha_0^\sigma) + g(t, \beta_1^\sigma) + g(t, \beta_0^\sigma) \\ &\quad - g_x(t, \alpha_0^\sigma)\,(\beta_0^\sigma - \beta_1^\sigma) - g(t, \beta_1^\sigma) \\ &= f(t, \alpha_1^\sigma) - \frac{1}{2!} f_{xx}(t, c_1)\,(\alpha_1^\sigma - \alpha_0^\sigma)^2 + g(t, \beta_1^\sigma) + g_x(t, c_2)\,(\beta_0^\sigma - \beta_1^\sigma) \\ &\quad - g_x(t, \alpha_0^\sigma)\,(\beta_0^\sigma - \beta_1^\sigma) \\ &\le f(t, \alpha_1^\sigma) + g(t, \beta_1^\sigma) + [g_x(t, c_2) - g_x(t, \alpha_0^\sigma)](\beta_0^\sigma - \beta_1^\sigma) \\ &= f(t, \alpha_1^\sigma) + g(t, \beta_1^\sigma) + g_{xx}(t, c_3)\,(c_2 - \alpha_0^\sigma)\,(\beta_0^\sigma - \beta_1^\sigma) \\ &\le f(t, \alpha_1^\sigma) + g(t, \beta_1^\sigma), \end{aligned}$$

where $\alpha_0^\sigma \le c_1, c_2, c_3 \le \beta_0^\sigma$, in view of the fact $f_{xx} \ge 0$ and $g_{xx} \le 0$.

Similarly we obtain

$$\begin{aligned}
-(p(t)\beta_1^{\Delta})^{\nabla} + q(t)\beta_1^{\sigma} &= f(t,\beta_0^{\sigma}) + g(t,\alpha_0^{\sigma}) + f_x(t,\alpha_0^{\sigma})\,(\beta_1^{\sigma}-\beta_0^{\sigma}) + g_x(t,\alpha_0^{\sigma})\,(\alpha_1^{\sigma}-\alpha_0^{\sigma}) \\
&= f(t,\beta_1^{\sigma}) + f(t,\beta_0^{\sigma}) - f_x(t,\alpha_0^{\sigma})\,(\beta_0^{\sigma}-\beta_1^{\sigma}) - f(t,\beta_1^{\sigma}) \\
&\quad + g(t,\alpha_0^{\sigma}) + g_x(t,\alpha_0^{\sigma})\,(\alpha_1^{\sigma}-\alpha_0^{\sigma}) \\
&= f(t,\beta_1^{\sigma}) + [f_x(t,c_1) - f_x(t,\alpha_0^{\sigma})](\beta_0^{\sigma}-\beta_1^{\sigma}) \\
&\quad + g(t,\alpha_1^{\sigma}) - \frac{1}{2!}g_{xx}(t,c_2)\,(\alpha_1^{\sigma}-\alpha_0^{\sigma})^2 \\
&\ge f(t,\beta_1^{\sigma}) + g(t,\alpha_1^{\sigma}) + f_{xx}(t,c_3)\,(c_1-\alpha_0^{\sigma})\,(\beta_0^{\sigma}-\beta_1^{\sigma}) \\
&\ge f(t,\beta_1^{\sigma}) + g(t,\alpha_1^{\sigma}),
\end{aligned}$$

where $\alpha_0^{\sigma} \le c_1, c_2, c_3 \le \beta_0^{\sigma}$, because of the fact $f_{xx} \ge 0$ and $g_{xx} \le 0$.

We can conclude from the above estimates that α_1 and β_1 are coupled lower and upper solutions of Type I, respectively, for the BVP (2.1) and (2.2), and it then follows from Theorem 2.1 that $\alpha_1(t) \le \beta_1(t)$, $t \in [\rho(a),\sigma(b)]$. Consequently, these results yield

$$\alpha_0(t) \le \alpha_1(t) \le \beta_1(t) \le \beta_0(t), \quad t \in [\rho(a),\sigma(b)].$$

Continuing this process by induction, we obtain sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with

$$\alpha_n(t) \le \alpha_{n+1}(t) \le \beta_{n+1}(t) \le \beta_n(t), \quad t \in [\rho(a),\sigma(b)], n = 0,1,2,\ldots,$$

where for each n, α_{n+1} is a solution of the BVP defined by

$$-(p(t)\alpha_{n+1}^{\Delta})^{\nabla} + q(t)\alpha_{n+1}^{\sigma} = F(t,\alpha_{n+1}^{\sigma},\beta_{n+1}^{\sigma};\alpha_n,\beta_n), \quad t \in [a,b],$$

and the boundary conditions (2.2), and β_{n+1} is a solution of the BVP defined by

$$-(p(t)\beta_{n+1}^{\Delta})^{\nabla} + q(t)\beta_{n+1}^{\sigma} = G(t,\beta_{n+1}^{\sigma},\alpha_{n+1}^{\sigma};\alpha_n,\beta_n), \quad t \in [a,b],$$

and the boundary conditions (2.2).

Hence by induction, we have

$$\alpha_0(t) \le \alpha_1(t) \le \cdots \le \alpha_n(t) \le \beta_n(t) \le \cdots \le \beta_1(t) \le \beta_0(t). \tag{3.3}$$

Since $[a,b]$ is compact, it follows that the convergence of each sequence, $\{\alpha_n\}$ or $\{\beta_n\}$, is uniform. We shall show that each of the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ converges to the solution of the BVP (2.1) and (2.2). In Atici and Guseinov [18], the Green's function $G(t,s)$, associated with (2.2) and the related homogeneous equation of (2.1), has been constructed and expressed by

$$G(t,s) = \frac{1}{D}\begin{cases}(\sigma(b)-t)\,(\tau_1(s-\rho(a))+\tau_2), & \rho(a) \le s \le t \le \sigma(b),\\ (\sigma(b)-s)\,(\tau_1(t-\rho(a))+\tau_2), & \rho(a) \le t \le s \le \sigma(b),\end{cases} \tag{3.4}$$

where $D = \tau_1(\sigma(b)-\rho(a))+\tau_2$. The authors have also shown the positive property of $G(t,s)$ in [18]. If $x(t)$ is the solution of the BVP (2.1) and (2.2), then

$$\begin{aligned}
x(t) &= \frac{1}{D}\left\{\frac{\tau_3 x(\eta)}{\sigma(b)}(\tau_2-\tau_1\rho(a))\right\}(\sigma(b)-t) + \frac{\tau_3 x(\eta)}{\sigma(b)}t \\
&\quad + \frac{1}{D}\left\{x_1(t) + \int_{\rho(a)}^{\sigma(b)} G(t,s)\,(f+g)\,(s,x^{\sigma}(s))\nabla s\right\},
\end{aligned} \tag{3.5}$$

where $x_1(t)$ is the solution of the BVP defined by

$$-(p(t)x_1^{\Delta})^{\nabla} + q(t)x_1^{\sigma} = 0, \quad t \in [a,b],$$

subject to the boundary conditions (2.2).

Using Green's function, we can write $\alpha_n(t)$ and $\beta_n(t)$ as follows:

$$\begin{aligned}\alpha_n(t) &= \frac{1}{D}\left\{\frac{\tau_3\alpha_n(\eta)}{\sigma(b)}(\tau_2-\tau_1\rho(a))\right\}(\sigma(b)-t)+\frac{\tau_3\alpha_n(\eta)}{\sigma(b)}t\\ &\quad+\frac{1}{D}\left\{x_1(t)+\int_{\rho(a)}^{\sigma(b)}G(t,s)F(s,\alpha_n^\sigma,\beta_n^\sigma;\alpha_{n-1},\beta_{n-1})\nabla s\right\},\\ \beta_n(t) &= \frac{1}{D}\left\{\frac{\tau_3\beta_n(\eta)}{\sigma(b)}(\tau_2-\tau_1\rho(a))\right\}(\sigma(b)-t)+\frac{\tau_3\beta_n(\eta)}{\sigma(b)}t\\ &\quad+\frac{1}{D}\left\{x_1(t)+\int_{\rho(a)}^{\sigma(b)}G(t,s)G(s,\beta_n^\sigma,\alpha_n^\sigma;\alpha_{n-1},\beta_{n-1})\nabla s\right\}.\end{aligned}$$

In view of (3.3), we can conclude that the sequences $\alpha_n(t)$ and $\beta_n(t)$ converge uniformly and monotonically to some functions $\alpha(t)$ and $\beta(t)$, respectively. Note that

$$F(s,\alpha_n^\sigma,\beta_n^\sigma;\alpha_{n-1},\beta_{n-1})\to f(s,\alpha^\sigma)+g(s,\beta^\sigma),$$

and

$$G(s,\beta_n^\sigma,\alpha_n^\sigma;\alpha_{n-1},\beta_{n-1})\to f(t,\beta^\sigma)+g(t,\alpha^\sigma).$$

Further, $\alpha(t)$ and $\beta(t)$ are solutions of the BVPs

$$-(p(t)\alpha^\Delta)^\nabla+q(t)\alpha^\sigma=f(t,\alpha^\sigma)+g(t,\beta^\sigma),\quad t\in[a,b],$$

and

$$-(p(t)\beta^\Delta)^\nabla+q(t)\beta^\sigma=f(t,\beta^\sigma)+g(t,\alpha^\sigma),\quad t\in[a,b],$$

with the boundary conditions (2.2), respectively. Now, by the repeated application of Theorem 2.1, we have $\alpha(t)=\beta(t)$, $t\in[\rho(a),\sigma(b)]$. Again, applying Theorem 2.2, we can see $\alpha(t)=x(t)=\beta(t)$, $t\in[\rho(a),\sigma(b)]$. Hence, $\{\alpha_n\}$ and $\{\beta_n\}$ converge uniformly and monotonically to the unique solution of the BVP (2.1) and (2.2).

Now, we should show that the convergence of each sequence, $\{\alpha_n\}$ or $\{\beta_n\}$, is quadratic. Let $x(t)$ denote the unique solution of the BVP (2.1) and (2.2). Applying Taylor's theorem and (A2), there exist c_1, c_2, c_3 such that $\alpha_n^\sigma\le c_1,c_2,c_3\le\beta_n^\sigma$, and

$$\begin{aligned}&-(p(x-\alpha_{n+1})^\Delta)^\nabla+q(x^\sigma-\alpha_{n+1}^\sigma)=f(t,x^\sigma)+g(t,x^\sigma)-F(t,\alpha_{n+1}^\sigma,\beta_{n+1}^\sigma;\alpha_n,\beta_n),\\ &\quad= f(t,x^\sigma)+g(t,x^\sigma)-f(t,\alpha_n^\sigma)-g(t,\beta_n^\sigma)-f_x(t,\alpha_n^\sigma)\,(\alpha_{n+1}^\sigma-\alpha_n^\sigma)-g_x(t,\alpha_n^\sigma)\,(\beta_{n+1}^\sigma-\beta_n^\sigma)\\ &\quad\le f(t,x^\sigma)-f(t,\alpha_n^\sigma)-f_x(t,\alpha_n^\sigma)\,(x^\sigma-\alpha_n^\sigma)+g(t,x^\sigma)-g(t,\beta_n^\sigma)+g_x(t,\alpha_n^\sigma)\,(\beta_n^\sigma-x^\sigma)\\ &\quad= \frac{1}{2!}f_{xx}(t,c_1)\,(x^\sigma-\alpha_n^\sigma)^2-g_x(t,c_1)\,(\beta_n^\sigma-x^\sigma)+g_x(t,\alpha_n^\sigma)\,(\beta_n^\sigma-x^\sigma)\\ &\quad\le \frac{1}{2!}f_{xx}(t,c_1)\,(x^\sigma-\alpha_n^\sigma)^2+g_{xx}(t,c_3)\,(\alpha_n^\sigma-\beta_n^\sigma)(\beta_n^\sigma-x^\sigma)\\ &\quad\le \frac{1}{2!}f_{xx}(t,c_1)\,(x^\sigma-\alpha_n^\sigma)^2-g_{xx}(t,c_3)[(x^\sigma-\alpha_n^\sigma)+(\beta_n^\sigma-x^\sigma)](\beta_n^\sigma-x^\sigma)\\ &\quad= \frac{1}{2}M_1(x^\sigma-\alpha_n^\sigma)^2+M_2[(x^\sigma-\alpha_n^\sigma)+(\beta_n^\sigma-x^\sigma)](\beta_n^\sigma-x^\sigma),\end{aligned}$$

where $|f_{xx}(t,x)|\le M_1$ and $|g_{xx}(t,x)|\le M_2$.

Since $\tau_1\alpha_{n+1}(\rho(a))-\tau_2\alpha_{n+1}^\Delta(\rho(a))=\tau_1x(\rho(a))-\tau_2x^\Delta(\rho(a))$, $\alpha_{n+1}(\sigma(b))-\tau_3\alpha_{n+1}(\eta)=x(\sigma(b))-\tau_3x(\eta)$ and (3.5), we obtain

$$\begin{aligned}(x-\alpha_{n+1})\,(t) &= \frac{1}{D}\left\{\frac{\tau_3(x-\alpha_{n+1})\,(\eta)}{\sigma(b)}(\tau_2-\tau_1\rho(a))\right\}(\sigma(b)-t)+\frac{\tau_3(x-\alpha_{n+1})\,(\eta)}{\sigma(b)}t\\ &\quad+\frac{1}{D}\left\{\int_{\rho(a)}^{\sigma(b)}G(t,s)[-(p(x-\alpha_{n+1})^\Delta)^\nabla+q(x^\sigma-\alpha_{n+1}^\sigma)]\nabla s\right\}.\end{aligned}$$

By taking into account the following inequality

$$\frac{1}{D\sigma(b)}(\tau_2-\tau_1\rho(a))(\sigma(b)-t)+\frac{t}{\sigma(b)}=\frac{\sigma(b)[\tau_2-\tau_1\rho(a)+\tau_1 t]}{\sigma(b)D}\le 1,$$

and taking the maximum over $[a, b]$, it follows that

$$\|x-\alpha_{n+1}\|\le\frac{L}{1-\tau_3}\left[\frac{1}{2}M_1\|x-\alpha_n\|^2+M_2\|\beta_n-x\|^2+M_2\|x-\alpha_n\|\|\beta_n-x\|\right], \tag{3.6}$$

where $L=\max_{t\in[a,\,b]}\frac{1}{D}\int_{\rho(a)}^{\sigma(b)}G(t,s)\nabla s$.

Similar to $\{\alpha_n(t)\}$, for $\{\beta_n(t)\}$, we have

$$\|\beta_{n+1}-x\|\le\frac{L}{1-\tau_3}\left[\frac{1}{2}M_2\|x-\alpha_n\|^2+M_1\|\beta_n-x\|^2+M_1\|x-\alpha_n\|\|\beta_n-x\|\right]. \tag{3.7}$$

Hence combing (3.6) and (3.7), we get

$$\|x-\alpha_{n+1}\|+\|\beta_{n+1}-x\|\le M[\|x-\alpha_n\|+\|\beta_n-x\|]^2,$$

where M is an appropriate positive constant.

Throughout the following theorems, we use $f^{(i)}(t,x)$ as the usual ith-order partial derivative of f with respect to x.

Theorem 3.2. *For the BVP (2.1) and (2.2), assume that*

(B1) *$\alpha_0(t)$ and $\beta_0(t)$ are coupled lower and upper solutions of Type I, for the BVP (2.1) and (2.2), respectively, with $\alpha_0(t)\le\beta_0(t)$ on $[\rho(a),\sigma(b)]$;*

(B2) *$f^{(i)}(t,x)$, $g^{(i)}(t,x)$ $(i=0,1,2,\ldots,k)$ exist and are continuous on $[\rho(a),\sigma(b)]\times[\alpha_0(t),\beta_0(t)]$, satisfying $f^{(1)}(t,x)<0$, $g^{(1)}(t,x)\ge 0$ and further, $f^{(k)}(t,x)\ge 0$, $g^{(k)}(t,x)\le 0$.*

Then there exist monotone sequences, $\{\alpha_n\}$ and $\{\beta_n\}$, converging uniformly on $[\rho(a),\sigma(b)]$ to the unique solution of the BVP (2.1) and (2.2). Moreover, the convergence is even-order k $(k\ge 2)$.

Proof. Define F_1 and G_1 by

$$\begin{aligned}F_1(t,x^\sigma,y^\sigma;\alpha_0,\beta_0)\ &=\ \sum_{i=0}^{k-1}\frac{1}{i!}f^{(i)}(t,\alpha_0^\sigma)\,(x^\sigma-\alpha_0^\sigma)^i+\sum_{i=0}^{k-2}\frac{1}{i!}g^{(i)}(t,\beta_0^\sigma)\,(y^\sigma-\beta_0^\sigma)^i\\&\quad+\frac{1}{(k-1)!}g^{(k-1)}(t,\alpha_0^\sigma)\,(y^\sigma-\beta_0^\sigma)^{(k-1)},\\G_1(t,x^\sigma,y^\sigma;\alpha_0,\beta_0)\ &=\ \sum_{i=0}^{k-2}\frac{1}{i!}f^{(i)}(t,\beta_0^\sigma)\,(x^\sigma-\beta_0^\sigma)^i+\frac{1}{(k-1)!}f^{(k-1)}(t,\alpha_0^\sigma)\,(x^\sigma-\beta_0^\sigma)^{(k-1)}\\&\quad+\sum_{i=0}^{k-1}\frac{1}{i!}g^{(i)}(t,\alpha_0^\sigma)\,(y^\sigma-\alpha_0^\sigma)^i.\end{aligned}$$

We shall consider the BVP defined by

$$-(p(t)x^\Delta)^\nabla+q(t)x^\sigma=F_1(t,x^\sigma,y^\sigma;\alpha_0,\beta_0),\quad t\in[a,b], \tag{3.8}$$

subject to the boundary conditions (2.2) and the BVP defined by

$$-(p(t)x^\Delta)^\nabla+q(t)x^\sigma=G_1(t,x^\sigma,y^\sigma;\alpha_0,\beta_0),\quad t\in[a,b], \tag{3.9}$$

subject to the boundary conditions (2.2).

Using the procedure employed in the previous theorem, we can find that there exist solutions, $\alpha_1(t)$ of the BVP (3.8) and (2.2) and $\beta_1(t)$ of the BVP (3.9) and (2.2), such that $\alpha_0(t)\le\alpha_1(t)\le\beta_1(t)\le\beta_0(t)$, $t\in[\rho(a),\sigma(b)]$. Again, by using the procedure successively, we can obtain a monotone sequence satisfying

$$\alpha_0(t)\le\alpha_1(t)\le\cdots\le\alpha_n(t)\le\beta_n(t)\le\cdots\beta_1(t)\le\beta_0(t),\quad t\in[\rho(a),\sigma(b)],$$

where α_n, β_n of the sequence are, respectively, a solution of the BVP defined by

$$-(p(t)x^{\Delta})^{\nabla} + q(t)x^{\sigma} = F_1(t, x^{\sigma}, y^{\sigma}; \alpha_{n-1}, \beta_{n-1}), \quad t \in [a, b],$$

subject to the boundary conditions (2.2), and a solution of the BVP defined by

$$-(p(t)x^{\Delta})^{\nabla} + q(t)x^{\sigma} = G_1(t, x^{\sigma}, y^{\sigma}; \alpha_{n-1}, \beta_{n-1}), \quad t \in [a, b],$$

subject to the boundary conditions (2.2).

Next, we can claim that $\{\alpha_n\}$ and $\{\beta_n\}$ converge on $[\rho(a), \sigma(b)]$ to the unique solution $x(t)$ of the BVP (2.1) and (2.2). The details are omitted to avoid repetition. Finally, we show the convergence of even-order $k \geq 2$. By the Taylor's theorem and (B2), there exists $\alpha_n^{\sigma} \leq c_1, c_2, c_3 \leq \beta_n^{\sigma}$, such that

$$\begin{aligned}
&-(p(x-\alpha_{n+1})^{\Delta})^{\nabla} + q(x^{\sigma} - \alpha_{n+1}^{\sigma}) = f(t, x^{\sigma}) + g(t, x^{\sigma}) - F(t, \alpha_{n+1}^{\sigma}, \beta_{n+1}^{\sigma}; \alpha_n, \beta_n), \\
&\quad = f(t, x^{\sigma}) + g(t, x^{\sigma}) - \sum_{i=0}^{k-2} \frac{1}{i!} f^{(i)}(t, \alpha_n^{\sigma})(\alpha_{n+1}^{\sigma} - \alpha_n^{\sigma})^i - \frac{1}{(k-1)!} f^{(k-1)}(t, \alpha_n^{\sigma})(\alpha_{n+1}^{\sigma} - \alpha_n^{\sigma})^{k-1} \\
&\qquad - \sum_{i=0}^{k-2} \frac{1}{i!} g^{(i)}(t, \beta_n^{\sigma})(\beta_{n+1}^{\sigma} - \beta_n^{\sigma})^i - \frac{1}{(k-1)!} g^{(k-1)}(t, \alpha_n^{\sigma})(\beta_{n+1}^{\sigma} - \beta_n^{\sigma})^{k-1} \\
&\quad = f(t, x^{\sigma}) - f(t, \alpha_{n+1}^{\sigma}) + \frac{1}{(k-1)!} f^{(k-1)}(t, c_1)(\alpha_{n+1}^{\sigma} - \alpha_n^{\sigma})^{k-1} \\
&\qquad - \frac{1}{(k-1)!} f^{(k-1)}(t, \alpha_n^{\sigma})(\alpha_{n+1}^{\sigma} - \alpha_n^{\sigma})^{k-1} + g(t, x^{\sigma}) - g(t, \beta_{n+1}^{\sigma}) \\
&\qquad + \frac{1}{(k-1)!} g^{(k-1)}(t, c_2)(\beta_{n+1}^{\sigma} - \beta_n^{\sigma})^{k-1} - \frac{1}{(k-1)!} g^{(k-1)}(t, \alpha_n^{\sigma})(\beta_{n+1}^{\sigma} - \beta_n^{\sigma})^{k-1} \\
&\quad \leq \frac{1}{(k-1)!} f^{(k)}(t, c_3)(c_1 - \alpha_n^{\sigma})(\alpha_{n+1}^{\sigma} - \alpha_n^{\sigma})^{k-1} + \frac{1}{(k-1)!} g^{(k)}(t, c_4)(c_2 - \alpha_n^{\sigma})(\beta_{n+1}^{\sigma} - \beta_n^{\sigma})^{k-1} \\
&\quad \leq M_3[(x^{\sigma} - \alpha_n^{\sigma})^k + (\beta_n^{\sigma} - x^{\sigma})^k + (x^{\sigma} - \alpha_n^{\sigma})\,(\beta_n^{\sigma} - x^{\sigma})^{k-1}],
\end{aligned} \tag{3.10}$$

where $M_3 = \max_{t \in [a,b]}\{\frac{1}{(k-1)!} f^{(k)}(t, x), \frac{1}{(k-1)!} g^{(k)}(t, x)\}$.

Similarly,

$$-(p(\beta_{n+1} - x)^{\Delta})^{\nabla} + q(\beta_{n+1}^{\sigma} - x^{\sigma}) \leq M_3[(x^{\sigma} - \alpha_n^{\sigma})^k + (\beta_n^{\sigma} - x^{\sigma})^k + (x^{\sigma} - \alpha_n^{\sigma})\,(\beta_n^{\sigma} - x^{\sigma})^{k-1}]. \tag{3.11}$$

On the same arguments, in view of (3.10) and (3.11), on taking maximum over the interval $[a, b]$, we have

$$\|x - \alpha_{n+1}\| + \|\beta_{n+1} - x\| \leq M[\|x - \alpha_n\| + \|\beta_n - x\|]^k,$$

where M is an appropriate positive constant. □

4. Coupled upper and lower solutions of Type II

In this section, we will discuss coupled upper and lower solutions of Type II, for the BVP (2.1) and (2.2), and state results similar to those in Sections 2 and 3. Once again, we shall only state results whose proofs can be obtained using analogous arguments.

Theorem 4.1. *For the BVP* (2.1) *and* (2.2)*, assume that*

(i) *$\alpha(t)$ and $\beta(t)$ are coupled lower and upper solutions of Type* II*, on $[\rho(a), \sigma(b)]$, respectively;*
(ii) *$f(t, x)$ is strictly increasing in x and $g(t, x)$ is decreasing in x for $t \in [a, b]$.*

Then $\alpha(t) \leq \beta(t)$ on $[\rho(a), \sigma(b)]$.

Theorem 4.2. *For the BVP* (2.1) *and* (2.2)*, assume that*

(A1*) *$\alpha_0(t)$ and $\beta_0(t)$ are coupled lower and upper solutions of Type* II*, respectively, with $\alpha_0(t) \leq \beta_0(t)$ on $[\rho(a), \sigma(b)]$;*

(A2*) $f, g \in C^2([\rho(a), \sigma(b)] \times [\alpha_0, \beta_0])$, *satisfy*

$$f_{xx}(t,x) \geq 0, \qquad g_{xx}(t,x) \leq 0 \quad \text{for } (t,x) \in [\rho(a), \sigma(b)] \times [\alpha_0, \beta_0].$$

Then there exist monotone sequences, $\{\alpha_n\}$ and $\{\beta_n\}$, such that

$$-(p(t)\alpha_n^{\Delta})^{\nabla} + q(t)\alpha_n^{\sigma} = f(t, \beta_{n-1}^{\sigma}) + g(t, \alpha_{n-1}^{\sigma}) + f_x(t, \beta_{n-1}^{\sigma})\,(\beta_n^{\sigma} - \beta_{n-1}^{\sigma}) + g_x(t, \beta_{n-1}^{\sigma})\,(\alpha_n^{\sigma} - \alpha_{n-1}^{\sigma}),\quad t \in [a,b]$$

and boundary conditions (2.2) hold for $\{\alpha_n\}$ and that

$$-(p(t)\beta_n^{\Delta})^{\nabla} + q(t)\beta_n^{\sigma} = f(t, \alpha_{n-1}^{\sigma}) + g(t, \beta_{n-1}^{\sigma}) + f_x(t, \beta_{n-1}^{\sigma})\,(\alpha_n^{\sigma} - \alpha_{n-1}^{\sigma}) + g_x(t, \beta_{n-1}^{\sigma})\,(\beta_n^{\sigma} - \beta_{n-1}^{\sigma}),\quad t \in [a,b]$$

and boundary conditions (2.2) hold for $\{\beta_n\}$. The two sequences converge uniformly and monotonically to the unique solution of the BVP (2.1) and (2.2) and the convergence is of order 2.

Proof. Analogous to the proof of Theorem 3.1. □

Theorem 4.3. *For the BVP (2.1) and (2.2), assume that*

(B1*) $\alpha_0(t)$ *and* $\beta_0(t)$ *are coupled lower and upper solutions of Type* II*, respectively, such that* $\alpha_0(t) \leq \beta_0(t)$ *on* $[\rho(a), \sigma(b)]$*;*

(B2*) $f^{(i)}(t,x)$, $g^{(i)}(t,x)$ $(i = 0, 1, 2, \ldots, k)$ *exist and are continuous on* $[\rho(a), \sigma(b)] \times [\alpha_0(t), \beta_0(t)]$*, satisfying* $f^{(1)}(t,x) > 0$, $g^{(1)}(t,x) \leq 0$ *and further,* $f^{(k)}(t,x) \geq 0$, $g^{(k)}(t,x) \leq 0$.

Then there exist monotone sequences, $\{\alpha_n\}$ and $\{\beta_n\}$, such that

$$\begin{aligned}-(p(t)\alpha_n^{\Delta})^{\nabla} + q(t)\alpha_n^{\sigma} &= \sum_{i=0}^{k-1} \frac{1}{i!} f^{(i)}(t, \beta_{n-1}^{\sigma})(\beta_n^{\sigma} - \beta_{n-1}^{\sigma})^i + \sum_{i=0}^{k-2} \frac{1}{i!} g^{(i)}(t, \alpha_{n-1}^{\sigma})(\alpha_n^{\sigma} - \alpha_{n-1}^{\sigma})^i \\ &\quad + \frac{1}{(k-1)!} g^{(k-1)}(t, \beta_{n-1}^{\sigma})(\alpha_n^{\sigma} - \alpha_{n-1}^{\sigma})^{k-1}, \quad t \in [a,b],\end{aligned}$$

and boundary conditions (2.2) hold for $\{\alpha_n\}$ and that;

$$\begin{aligned}-(p(t)\beta_n^{\Delta})^{\nabla} + q(t)\beta_n^{\sigma} &= \sum_{i=0}^{k-2} \frac{1}{i!} f^{(i)}(t, \alpha_{n-1}^{\sigma})\,(\alpha_n^{\sigma} - \alpha_{n-1}^{\sigma})^i + \frac{1}{(k-1)!} f^{(k-1)}(t, \beta_{n-1}^{\sigma})(\alpha_n^{\sigma} - \alpha_{n-1}^{\sigma})^{k-1} \\ &\quad + \sum_{i=0}^{k-1} \frac{1}{i!} g^{(i)}(t, \beta_{n-1}^{\sigma})\,(\beta_n^{\sigma} - \beta_{n-1}^{\sigma})^i, \quad t \in [a,b],\end{aligned}$$

and boundary conditions (2.2) hold for $\{\beta_n\}$. Both sequences converge uniformly on $[\rho(a), \sigma(b)]$ to the unique solution of the BVP (2.1) and (2.2). Moreover, the convergence is even-order k $(k \geq 2)$.

Proof. Analogous to the proof of Theorem 3.2. □

Remark 4.1. Similar results can be obtained for the other coupled upper and lower solutions [13] of the BVP (2.1) and (2.2).

Example 4.1. Let us consider the following BVP

$$\begin{aligned}&-x^{\Delta\nabla} + x^{\sigma} = x^{\sigma 2} - x^{\sigma}, \quad t \in [0,1], \\ &x(\rho(0)) = x^{\Delta}(\rho(0)), \qquad x(\sigma(1)) = \frac{1}{2}x(\eta),\end{aligned} \tag{4.1}$$

where $f(t,x) = x^2 - x$.

If we choose $\alpha_0(t) \equiv -\frac{1}{2}$, $\beta_0(t) \equiv \frac{1}{2}$, and $0 \le t \le 1$, we get

$$\begin{aligned}
&-\frac{1}{2} = -\alpha_0^{\Delta\nabla} + \alpha_0^{\sigma} \le \frac{3}{4},\\
&\alpha_0(\rho(0)) \le \alpha_0^{\Delta}(\rho(0)), \qquad \alpha_0(\sigma(1)) \le \frac{1}{2}\alpha_0(\eta);\\
&\frac{1}{2} = -\beta_0^{\Delta\nabla} + \beta_0^{\sigma} \ge -\frac{1}{4},\\
&\beta_0(\rho(0)) \ge \beta_0^{\Delta}(\rho(0)), \qquad \beta_0(\sigma(1)) \ge \frac{1}{2}\beta_0(\eta).
\end{aligned}$$

Thus $\alpha_0(t) \equiv -\frac{1}{2}$ and $\beta_0(t) \equiv \frac{1}{2}$ are coupled lower and upper solutions of Type I for the BVP (4.1), respectively. Also, by Theorem 2.1, there is a solution in $[\alpha_0, \beta_0]$ for $t \in [0, 1]$. As in the proof of Theorem 3.1, for each n, α_{n+1} is a solution the BVP defined by

$$-x^{\Delta\nabla} + x^{\sigma} = f(t, \alpha_n^{\sigma}) + f_x(t, \alpha_n^{\sigma})\,(x^{\sigma} - \alpha_n^{\sigma}), \quad t \in [0, 1],$$

and the boundary conditions (2.2), and β_{n+1} is a solution of the BVP defined by

$$-x^{\Delta\nabla} + x^{\sigma} = f(t, \beta_n^{\sigma}) + f_x(t, \alpha_n^{\sigma})\,(x^{\sigma} - \beta_n^{\sigma}), \quad t \in [0, 1],$$

and the boundary conditions (2.2).

One can easily see that α_n and β_n are obtained by

$$\alpha_n^{\sigma} = \frac{-{\alpha_{n-1}^{\sigma}}^2}{2 - 2\alpha_{n-1}^{\sigma}}, \qquad \beta_n^{\sigma} = \frac{\beta_{n-1}^{\sigma}(\beta_{n-1}^{\sigma} - 2\alpha_{n-1}^{\sigma})}{2 - 2\alpha_{n-1}^{\sigma}}.$$

Thus, it follows that there are monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$ converging uniformly to the unique solution, which is zero, of the above BVP (4.1). Therefore, we can apply Theorem 3.1 to the BVP (4.1).

Example 4.2. Let us discuss the following BVP

$$\begin{aligned}
&-x^{\Delta\nabla} + 2x^{\sigma} = -x^{\sigma 2} - 2x^{\sigma}, \quad t \in [0, 1],\\
&x(\rho(0)) = x^{\Delta}(\rho(0)), \qquad x(\sigma(1)) = \frac{1}{2}x(\eta),
\end{aligned} \tag{4.2}$$

where $g(t, x) = -x^2 - 2x$.

Choose $\alpha_0(t) \equiv -\frac{1}{2}$, $\beta_0(t) \equiv 1$, and $0 \le t \le 1$, we have

$$\begin{aligned}
&-1 = -\alpha_0^{\Delta\nabla} + 2\alpha_0^{\sigma} \le \frac{3}{4},\\
&\alpha_0(\rho(0)) \le \alpha_0^{\Delta}(\rho(0)), \qquad \alpha_0(\sigma(1)) \le \frac{1}{2}\alpha_0(\eta);\\
&2 = -\beta_0^{\Delta\nabla} + 2\beta_0^{\sigma} \ge -3,\\
&\beta_0(\rho(0)) \ge \beta_0^{\Delta}(\rho(0)), \qquad \beta_0(\sigma(1)) \ge \frac{1}{2}\beta_0(\eta).
\end{aligned}$$

Then $\alpha_0(t) \equiv -\frac{1}{2}$ and $\beta_0(t) \equiv 1$ are coupled lower and upper solutions of Type II for the BVP (4.2), respectively. Also, in view of Theorem 2.1, we can conclude that there is a solution in $[-1/2, 1]$ for $t \in [0, 1]$. Moreover, applying the given iterates of Theorem 4.2, for each n, α_{n+1} is a solution of the BVP defined by

$$-x^{\Delta\nabla} + 2x^{\sigma} = g(t, \alpha_n^{\sigma}) + g_x(t, \beta_n^{\sigma})\,(x^{\sigma} - \alpha_n^{\sigma}), \quad t \in [0, 1],$$

and boundary conditions (2.2), and β_{n+1} is a solution of the BVP defined by

$$-x^{\Delta\nabla} + 2x^{\sigma} = g(t, \beta_n^{\sigma}) + g_x(t, \beta_n^{\sigma})\,(x^{\sigma} - \beta_n^{\sigma}), \quad t \in [0, 1],$$

Table 1
Table of two α, β-iterates of (4.3)

t	$\alpha_1(t)$	$\alpha_2(t)$	$\beta_2(t)$	$\beta_1(t)$
0.1	0.0888	0.0890	0.0890	0.1221
0.2	0.1712	0.1722	0.1723	0.2291
0.3	0.2504	0.2530	0.2534	0.3252
0.4	0.3296	0.3344	0.3351	0.4143
0.5	0.4120	0.4190	0.4200	0.5000
0.6	0.5009	0.5096	0.5106	0.5857
0.7	0.5999	0.6087	0.6095	0.6748
0.8	0.7129	0.7191	0.7196	0.7709
0.9	0.8445	0.8455	0.8456	0.8779

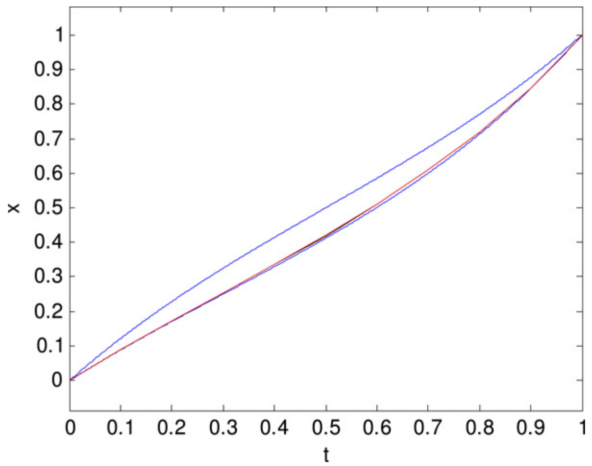

Fig. 1. The curves of the iterates, where α_1 and β_1 are blue lines, α_2 is black line, and β_2 is red line. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

and the boundary conditions (2.2).

Clearly, α_n and β_n are obtained by

$$\alpha_n^\sigma = \frac{\alpha_{n-1}^\sigma(2\beta_{n-1}^\sigma - \alpha_{n-1}^\sigma)}{4 - 2\beta_{n-1}^\sigma}, \qquad \beta_n^\sigma = \frac{{\beta_{n-1}^\sigma}^2}{4 - 2\beta_{n-1}^\sigma}.$$

We can see that there are monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$ converging uniformly to the unique solution, which is $x = 0$, of the above BVP (4.2). Therefore, Theorem 4.2 is applicable to the BVP (4.2).

Example 4.3. Let $\mathbb{T} = \mathbb{R}$. Consider the following BVP

$$\begin{aligned} &-x'' = x^3 - 4x + 1, \quad t \in [0, 1], \\ &x(0) = 0, \quad x(1) = 1, \end{aligned} \tag{4.3}$$

where $f(t, x) = x^3 - 4x + 1$.

It is easy to check that $\alpha_0(t) \equiv 0$ and $\beta_0(t) \equiv 1$ are coupled lower and upper solutions of Type I for the BVP (4.3), respectively. Also, by Theorem 2.2, there is a solution in [0,1] for $t \in [0, 1]$. Hence we can apply Theorem 3.1 to the BVP (4.3). With the help of Mathematica, we can derive the α, β-iterates in Table 1. The α-iterates and the β-iterates can be seen on Fig. 1, where α_1 and β_1 are blue lines.

Table 2
Table of two α, β-iterates of (4.4)

t	$\alpha_1(t)$	$\alpha_2(t)$	$\beta_2(t)$	$\beta_1(t)$
0.1571	0.2614	0.3746	0.3796	0.4656
0.3142	0.4678	0.6331	0.6512	0.8168
0.4712	0.6345	0.8475	0.8751	1.0800
0.6283	0.7740	1.0246	1.0530	1.2746
0.7854	0.8967	1.1682	1.1906	1.4152
0.9425	1.0117	1.2832	1.2971	1.5123
1.0996	1.1275	1.3754	1.3818	1.5730
1.2566	1.2529	1.4503	1.4521	1.6019
1.4137	1.3970	1.5134	1.5136	1.6012

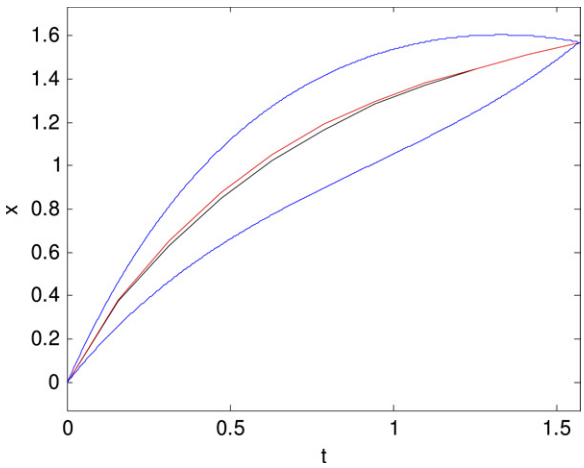

Fig. 2. The curves of the iterates, where α_1 and β_1 are blue lines, α_2 is black line, and β_2 is red line. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Example 4.4. Let $\mathbb{T} = \mathbb{R}$. Consider the following BVP

$$\begin{aligned} &-x'' = 3\cos x, \qquad t \in \left[0, \frac{\pi}{2}\right], \\ &x(0) = 0, \qquad x\left(\frac{\pi}{2}\right) = \frac{\pi}{2}, \end{aligned} \tag{4.4}$$

where $g(t, x) = 3\cos x$.

One can easily that $\alpha_0(t) \equiv 0$ and $\beta_0(t) \equiv \frac{\pi}{2}$ are coupled lower and upper solutions of Type II for the BVP (4.4), respectively. Also, by Theorem 2.2, there is a solution in $[0, \frac{\pi}{2}]$ for $t \in [0, \frac{\pi}{2}]$. Hence we can apply Theorem 4.1 to the BVP (4.4). With the help of Mathematica, we can derive the α, β-iterates in Table 2. The α-iterates and the β-iterates can be seen on Fig. 2, where α_1 and β_1 are blue lines.

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